

Techniques for the Cograph Editing Problem: Module Merge is equivalent to Editing P_4 's

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Cographs are graphs in which no four vertices induce a simple connected path P_4 . Cograph editing is to find for a given graph $G = (V, E)$ a set of at most k edge additions and deletions that transform G into a cograph. This combinatorial optimization problem is NP-hard. It has, recently found applications in the context of phylogenetics, hence good heuristics are of practical importance.

It is well-known that the cograph editing problem can be solved independently on the so-called strong prime modules of the modular decomposition of G . We show here that editing the induced P_4 's of a given graph is equivalent to resolving strong prime modules by means of a newly defined merge operation \boxplus on the submodules. This observation leads to a new exact algorithm for the cograph editing problem that can be used as a starting point for the construction of novel heuristics.

Keywords: Cograph Editing, Module Merge, Twin Relation, Strong Prime Modules

1 Introduction

Cographs are among the best-studied graph classes. In particular the fact that many problems that are NP-complete for arbitrary graphs become polynomial-time solvable on cographs [CPS85, BLS99, GHN13] makes them an attractive starting point for constructing heuristics. As noted already in [CLSB81], the input for several combinatorial optimization problems, such as exam scheduling or several variants of clustering problems, is naturally expected to have few induced P_4 s. Since graphs without an induced P_4

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are exactly the cographs, identifying the closest cograph and solving the problem at hand for the modified input becomes a viable strategy.

It was recently shown that *orthology*, a key concept in evolutionary biology in phylogenetics, is intimately tied to cographs. Two genes in a pair of related species are said to be orthologous if their last common ancestor was a speciation event. The orthology relation on a set of genes forms a cograph [HHRH⁺13]. This relation can be estimated directly from biological sequence data, albeit in a necessarily noisy form. Correcting such an initial estimate to the nearest cograph, i.e., cograph-editing, thus has recently become an computational problem of considerable practical interest in computational biology [HWL⁺15]. However, the (decision version of the) problem to edit a given graph into a cograph is NP-complete [LWGC11, LWGC12]. We showed in [HWL⁺15] that the cograph-editing problem is amenable to formulations as Integer Linear Programs (ILP). Computational experiments showed, however, that the performance of the ILP scales not very favorably, thus limiting exact ILP solutions in practice to moderate-sized data. Fast and accurate heuristics for cograph-editing are therefore of immediate practical interest in the field of phylogenomics.

The cotree of a cograph coincides with the modular decomposition tree [Gal67], which is defined for all graphs. We investigate here how edge editing on an approximate cograph is related to editing the corresponding modular decomposition trees.

2 Basic Definitions

We consider simple finite undirected graphs $G = (V, E)$ without loops. The notation $G + e$, $G - e$ and $G \triangle e$ is used to denote the graph $(V, E \cup \{e\})$, $(V, E \setminus \{e\})$ and $(V, E \triangle \{e\})$, respectively. A graph $H = (W, E')$ is a *subgraph* of a graph $G = (V, E)$, in symbols $H \subseteq G$, if $W \subseteq V$ and $E' \subseteq E$. If $H \subseteq G$ and $xy \in E'$ if and only if $xy \in E$ for all $x, y \in W$, then H is called an *induced* subgraph. We often denote such an induced subgraph $H = (W, E')$ by $G[W]$. A *connected component* of G is a connected induced subgraph that is maximal w.r.t. inclusion. The complement \bar{G} of a graph $G = (V, E)$ has vertex set V and edge set $E(\bar{G}) = \{xy \mid x, y \in V, x \neq y, xy \notin E\}$. The *complete graph* $K_{|V|} = (V, E)$ has edge set $E = \binom{V}{2}$. We write $G \simeq H$ for two isomorphic graphs G and H .

Let $G = (V, E)$ be a graph. The (*open*) *neighborhood* $N(v)$ is defined as $N(v) = \{x \mid vx \in E\}$. The (*closed*) *neighborhood* $N[v]$ is then $N[v] = N(v) \cup \{v\}$. If there is a risk of confusion we will write $N_G(v)$, resp., $N_G[v]$ to indicate that the respective neighborhoods are taken w.r.t. G . The *degree* $\deg(v)$ of a vertex is defined as $\deg(v) = |N(v)|$.

A *tree* is a connected graph that does not contain cycles. A *path* is a tree where every vertex has degree 1 or 2. A *rooted tree* $T = (V, E)$ is a tree with one distinguished vertex $\rho \in V$. The first inner vertex $\text{lca}(x, y)$ that lies on both unique paths from two vertices x , resp., y to the root, is called *lowest common ancestor* of x and y . It is well-known that there is a one-to-one correspondence between (isomorphism classes of) rooted trees on V and so-called hierarchies on V . For a finite set V , a *hierarchy on V* is a subset \mathcal{C} of the power set $\mathcal{P}(V)$ such that (i) $V \in \mathcal{C}$, (ii) $\{x\} \in \mathcal{C}$ for all $x \in V$ and (iii) $p \cap q \in \{p, q, \emptyset\}$ for all $p, q \in \mathcal{C}$.

Theorem 2.1 ([SS03]) *Let \mathcal{C} be a collection of non-empty subsets of V . Then, there is a rooted tree $T = (W, E)$ on V with $\mathcal{C} = \{L(v) \mid v \in W\}$ if and only if \mathcal{C} is a hierarchy on V .*

3 Cographs, P_4 -sparse Graphs and the Modular Decomposition

3.1 Introduction to Cographs

Cographs are defined as the class of graphs formed from a single vertex under the closure of the operations of union and complementation, namely: (i) a single-vertex graph K_1 is a cograph; (ii) the disjoint union $G = (V_1 \cup V_2, E_1 \cup E_2)$ of cographs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a cograph; (iii) the complement \bar{G} of a cograph G is a cograph. The name cograph originates from *complement reducible graphs*, as by definition, cographs can be “reduced” by stepwise complementation of connected components to totally disconnected graphs [Sei74].

It is well-known that for each induced subgraph H of a cograph G either H is disconnected or its complement \bar{H} is disconnected [BLS99]. This, in particular, allows representing the structure of a cograph $G = (V, E)$ in an unambiguous way as a rooted tree $T = (W, F)$, called *cotree*: If the considered cograph is the single vertex graph K_1 , then output the tree $(\{u\}, \emptyset)$. Else if the given cograph G is connected, create an inner vertex u in the cotree with label “series”, build the complement \bar{G} and add the connected components of \bar{G} as children of u . If G is not connected, then create an inner vertex u in the cotree with label “parallel” and add the connected components of G as children of u . Proceed recursively on the respective connected components that consists of more than one vertex. Eventually, this cotree will have leaf-set $V \subseteq W$ and inner vertices $u \in W \setminus V$ are labeled with either “parallel” or “series” s.t. $xy \in E$ if and only if $u = \text{lca}_T(x, y)$ is labeled “series”. Since a cograph and its cotree are uniquely determined by each other, one can use the cotree representation to test in linear-time whether two cographs are isomorphic [CPS85].

The complement of a path on four vertices P_4 is again a P_4 and hence, such graphs are not cographs. Intriguingly, cographs have indeed a quite simple characterization as P_4 -free graphs, that is, no four vertices induce a P_4 . A number of further equivalent characterizations are given in [BLS99] and Theorem 4. Determining whether a graph is a cograph can be done in linear time [CPS85, BHP08].

3.2 Modules and the Modular Decomposition

The concept of *modular decompositions* (MD) is defined for arbitrary graphs G . It presents the structure of G in the form of a tree that generalizes the idea of cotrees. However, in general much more information needs to be stored at the inner vertices of this tree if the original graph is to be recovered.

The MD is based on modules, which are also known as autonomous sets [MR84, Möh85], closed sets [Gal67], clan [EGMS94], stable sets, clumps [Bla78] or externally related sets [HM79]. A *module* of a given graph $G = (V, E)$ is a subset $M \subseteq V$ with the property that for all vertices $x, y \in M$ holds that $N(y) \setminus M = N(x) \setminus M$. The vertices within a given module M are therefore not distinguishable by the part of their neighborhoods that lie “outside” M . Modules can thus be seen as generalization of the notion of connected components. We denote with $\mathbb{M}(G)$ the set of all modules of $G = (V, E)$. Clearly, the vertex set V and the singletons $\{v\}$, $v \in V$ are modules, called *trivial* modules. A graph G is called *prime* if it only contains trivial modules. For a module M of G and a vertex $v \in M$, we define the out_M -neighborhood of v in G as the set $N_G^M(v) := N_G(v) \setminus M$. Since for each $v, w \in M$ the out_M -neighborhoods are identical, $N_G^M(v) = N_G^M(w)$, we can equivalently define the out_M -neighborhood of the module M as $N_G^M := N_G^M(v)$, $v \in M$.

For a graph $G = (V, E)$ let M and M' be disjoint subsets of V . We say that M and M' are adjacent (in G) if each vertex of M is adjacent to all vertices of M' ; the sets are non-adjacent if none of the vertices of M is adjacent to a vertex of M' . Two disjoint modules are either adjacent or non-adjacent [Möh85].

One can therefore define the *quotient graph* G/\mathbb{M}' for an arbitrary subset $\mathbb{M}' \subseteq \mathbb{M}(G)$ of pairwise disjoint modules: G/\mathbb{M}' has \mathbb{M}' as its vertex set and $(M_i, M_j) \in E(G/\mathbb{M}')$ if and only if M_i and M_j are adjacent in G .

A module M is called *strong* if for any module $M' \neq M$ either $M \cap M' = \emptyset$, or $M \subseteq M'$, or $M' \subseteq M$, i.e., a strong module does not overlap any other module. The set of all strong modules $\text{MD}(G) \subseteq \mathbb{M}(G)$ thus forms a hierarchy, the so-called *modular decomposition* of G . While arbitrary modules of a graph form a potentially exponential-sized family, however, the sub-family of strong modules has size $O(|V(G)|)$ [HDMP04].

Let $\mathbb{P} = \{M_1, \dots, M_k\}$ be a partition of the vertex set of a graph $G = (V, E)$. If every $M_i \in \mathbb{P}$ is a module of G , then \mathbb{P} is a modular partition of G . A non-trivial modular partition $\mathbb{P} = \{M_1, \dots, M_k\}$ that contains only maximal (w.r.t inclusion) strong modules is a maximal modular partition. We denote the (unique) maximal modular partition of G by $\mathbb{P}_{\max}(G)$. We will refer to the elements of $\mathbb{P}_{\max}(G[M])$ as the *children of M* . This terminology is motivated by the following considerations:

The hierarchical structure of $\text{MD}(G)$ gives rise to a canonical tree representation of G , which is usually called the *modular decomposition tree* $\text{MDT}(G)$ [MR84, HP10]. The root of this tree is the trivial module V and its $|V|$ leaves are the trivial modules $\{v\}$, $v \in V$. The set of leaves L_v associated with the subtree rooted at an inner vertex v induces a strong module of G . Moreover, inner vertices v are labeled “parallel” if the induced subgraph $G[L_v]$ is disconnected, “series” if the complement $\overline{G}[L_v]$ is disconnected and “prime” otherwise, i.e., if $G[L_v]$ and $\overline{G}[L_v]$ are both connected. The module L_v of the induced subgraph $G[L_v]$ associated to a vertex v labeled “prime” is called prime module. Note, the latter does not imply that $G[L_v]$ is prime, however, in all cases $G[L_v]/\mathbb{P}_{\max}(G[L_v])$ is prime [HP10]. Similar to cotrees it holds that $xy \in E$ if $u = \text{lca}_{\text{MDT}(G)}(xy)$ is labeled “series”, and $xy \notin E$ if $u = \text{lca}_{\text{MDT}(G)}(xy)$ is labeled “parallel”. However, to trace back the full structure of a given graph G from $\text{MDT}(G)$ one has to store additionally the information of the subgraph $G[L_v]/\mathbb{P}_{\max}(G[L_v])$ in the vertices v labeled “prime”. Although, $\text{MD}(G) \subseteq \mathbb{M}(G)$ does not represent all modules, we state the following remarkable fact [Möh85, DGM97]: Any subset $M \subseteq V$ is a module if and only if $M \in \text{MD}(G)$ or M is the union of children of non-prime modules. Thus, $\text{MDT}(G)$ represents at least implicitly all modules of G .

A simple polynomial time recursive algorithm to compute $\text{MDT}(G)$ is as follows [HP10]: (1) compute the maximal modular partition $\mathbb{P}_{\max}(G)$; (2) label the root node according to the parallel, series or prime type of G ; (3) for each strong module M of \mathbb{P}_{\max} , compute $\text{MDT}(G[M])$ and attach it to the root node. The first polynomial algorithm to compute the modular decomposition is due to Cowan *et al.* [CJS72], and it runs in $O(|V|^4)$. Improvements are due to Habib and Maurer [HM79], who proposed a cubic time algorithm, and to Müller and Spinrad [MS89], who designed a quadratic time algorithm. The first two linear time algorithms appeared independently in 1994 [CH94, MS94]. Since then a series of simplified algorithms has been published, some running in linear time [DGM01, MS99, TCHP08], and others in almost linear time [DGM01, MS00, HPV99, HDMP04].

We give here two simple lemmata for further reference.

Lemma 3.1 *Let M be a module of a graph $G = (V, E)$ and $M' \subseteq M$. Then M' is a module of $G[M]$ if and only if M' is a module of G .*

Furthermore, suppose $M \in \text{MD}(G)$ is strong module of G . Then M' is a strong module of $G[M]$ if and only if M' is a strong module of G .

Proof: Let $M \in \mathbb{M}(G)$. If M' is a module of $G[M]$, then all $x, y \in M'$ have the same $\text{out}_{M'}$ -neighbors

in $G[M]$. Since M is a module of G and $M' \subseteq M$, for all $x, y \in M'$ the $\text{out}_{M'}$ -neighborhood and out_M -neighborhood in $G[V \setminus M]$ are identical. Thus, all $x, y \in M'$ have the same $\text{out}_{M'}$ -neighborhood in G .

If $M' \subseteq M$ is a module in G then, in particular, the $\text{out}_{M'}$ -neighborhood in $G[M]$ must be identical for all $x, y \in M'$, and thus M' is a module in $G[M]$.

Let $M \in \text{MD}(G)$ and assume that M' is a strong module of $G[M]$. Since M is a strong module in G it does not overlap any other modules in G . Assume for contradiction that M' is not a strong module of G . Hence M' must overlap some module M'' in G . This module M'' cannot be entirely contained in M as otherwise, M'' and M' overlap in $G[M]$ implying that M' is not a strong module of $G[M]$, a contradiction. But then M and M'' must overlap, contradicting that $M \in \text{MD}(G)$.

If M' is a strong module of G then it does not overlap any module of G . As every module of G is also a module of $G[M]$ (and vice versa) it follows that M' does not overlap any module of $G[M]$ and thus, M' must be a strong module of $G[M]$. \square

Lemma 3.2 *Let G be an arbitrary graph and G' be a cograph on the same vertex set V so that $\mathbb{M}(G) \subset \mathbb{M}(G')$, i.e., every module of G is a module of G' . Moreover, let $\mathbb{P}_{\max} := \mathbb{P}_{\max}(G)$ be the maximal modular partition of G . Then \mathbb{P}_{\max} is a modular partition of G' and G'/\mathbb{P}_{\max} is a cograph.*

Proof: Since $\mathbb{M}(G) \subset \mathbb{M}(G')$ we can immediately conclude that \mathbb{P}_{\max} is a (not necessarily maximal) modular partition of G' and therefore the quotient G'/\mathbb{P}_{\max} is well-defined. Assume, for contradiction, that G'/\mathbb{P}_{\max} is not a cograph. Then G'/\mathbb{P}_{\max} must contain one induced P_4 , say $M_1 - M_2 - M_3 - M_4$. As M_1, \dots, M_4 are modules of G' and since two disjoint modules are either adjacent or non-adjacent it follows that G' must contain an induced P_4 of the form $x_1 - x_2 - x_3 - x_4$ with $x_i \in M_i$, $1 \leq i \leq 4$, a contradiction. \square

3.3 The Twin-Relation

A special kind of module that will play a central role in this contribution are *twins*. Two vertices $x, y \in V$ are called twins if $\{x, y\}$ is a module of G . Twins $x, y \in V$ are called *true twins* if $xy \in E$ and *false twins* otherwise. Twins x and y therefore satisfy $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. In particular, for true twins $\{x, y\}$ we can infer that $N[x] = N[y]$ and for false twins $\{x, y\}$ we only have $N(x) = N(y)$.

Definition 1 *Let $G = (V, E)$ be an arbitrary graph. The twin relation \mathcal{T} is the binary relation on V that contains all pairs of twins: $(x, y) \in \mathcal{T}$ if and only if x, y are twins in G . The pair $(x, x) \in \mathcal{T}$ is called trivial twin.*

Unless explicitly stated, we will use the phrase “a pair of twins” or “twins”, for short, to refer only to non-trivial twins.

Proposition 3.3 *Let $G = (V, E)$ be a given graph and \mathcal{T} the twin relation on V . Then the following statements hold:*

1. *The relation \mathcal{T} is an equivalence relation.*
2. *For every equivalence class $M \sqsubseteq \mathcal{T}$ the distinct elements of M are either all true or false twins.*
3. *Every equivalence class $M \sqsubseteq \mathcal{T}$ is a module of G and there is no other non-trivial strong module contained in M .*

Proof: We first prove Statement 1. and 2.: Clearly, \mathcal{T} is reflexive and symmetric. It remains to show that \mathcal{T} is transitive. Assume that $(x, y), (x, z) \in \mathcal{T}$. We show first that x, y and x, z can only be either true or false twins. Assume for contradiction that x, y are false twins and x, z are true twins. Hence, $(x, z) \in E(G)$ and since $N(x) = N(y)$ we have $(y, z) \in E(G)$. However, since $y \in N[z] = N[x]$ we have $y \in N(x)$ and hence $(x, y) \in E(G)$, a contradiction.

Let $(x, y), (x, z) \in \mathcal{T}$ be both false twins. Thus, $N(z) = N(x) = N(y)$, which implies that $(y, z) \in \mathcal{T}$. Moreover, since $N(z) = N(y)$, the vertices y and z cannot be adjacent and thus, x and z are false twins. Now assume that $(x, y), (x, z) \in \mathcal{T}$ are both true twins. Hence, $N[z] = N[x] = N[y]$ and, thus, $(y, z) \in E(G)$. Therefore, $(y, z) \in \mathcal{T}$ are true twins.

Thus, T is transitive and hence, an equivalence relation, where each equivalence class $M \sqsubseteq \mathcal{T}$ comprises either only false or only true twins.

Now, we prove Statement 3.: Statement 1. and 2. imply that $M \sqsubseteq \mathcal{T}$ contains either only true or false twins. If M contains only false twins, then $N(x) \setminus M = N(x) = N(y) = N(y) \setminus M$ for all $x, y \in M$ and thus, M is a module. If M contains only true twins x, y , then $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. If there is an additional vertex $z \in M$ we must have $N(x) \setminus \{y, z\} = N(y) \setminus \{x, z\}$. Induction on the number of elements of M shows that $N(x) \setminus M = N(y) \setminus M$ holds for all $x, y \in M \sqsubseteq \mathcal{T}$. Therefore, M is a module.

Finally, assume there is a non-trivial strong module M' contained in $M \sqsubseteq \mathcal{T}$. Let $x \in M' \subsetneq M$ and $z \in M \setminus M'$. Since $x, z \in M \sqsubseteq \mathcal{T}$, they are twins. Thus, $\{x, z\}$ is a module. Hence, M' cannot be strong module since $\{x, z\} \cap M' = \{x\}$ and thus, $\{x, z\}$ and M' overlap. □

Note that equivalence classes of \mathcal{T} are not necessarily strong modules, as the following example shows. Consider the graph $G = (\{0, 1, 2, 3\}, \{(2, 3)\})$. The twin relation \mathcal{T} on G has equivalence classes $\{0, 1\}$ and $\{2, 3\}$. However, $M = \{1, 2, 3\}$ is also a module, as $N(i) \setminus M = \emptyset$, $1 \leq i \leq 3$. In this case, M overlaps $\{0, 1\}$. The modular decomposition $\text{MD}(G)$ is $\{\{0\}, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{0, 1, 2, 3\}\}$, while $\mathbb{M}(G) \setminus \text{MD}(G) = \{\{0, 2, 3\}, \{1, 2, 3\}\}$.

3.4 P_4 -sparse Graphs and Spiders

Although the cograph-editing problem is NP-complete, it can be solved in polynomial time for so-called P_4 -sparse graphs [LWGC11, LWGC12], in which every set of five vertices induces at most one P_4 [Hoà85]. The efficient recognition of P_4 -sparse graphs is intimately connected to so-called spider graphs, a very peculiar class of prime graphs.

Lemma 3.4 [JO89, JO92]. *A graph G is P_4 sparse if and only if exactly one of the following three alternatives is true for every induced subgraph H of G : (i) H is not connected, (ii) \overline{H} is not connected, or (iii) H is a spider.*

Spiders come in two sub-types, called *thin* and *thick* [JO92, NG12]. A graph G is a *thin spider* if its vertex set can be partitioned into three sets K , S , and R so that (i) K is a clique; (ii) S is a stable set; (iii) $|K| = |S| \geq 2$; (iv) every vertex in R is adjacent to all vertices of K and none of the vertices of S ; and (v) each vertex in K is connected to exactly one vertex in S by an edge and *vice versa*. A graph G is a *thick spider* if its complement \overline{G} is a thin spider. The sets K , S , and R are usually referred to as the *body*, the set of *legs*, and *head*, resp., of a thin spider. The path P_4 is the only graph that is both a thin and thick spider. Interestingly, spider graphs are fully characterized by its degree sequences [BCF⁺15].

Lemma 3.4 in particular implies that any strong module $M \in \text{MD}(G)$ of a P_4 -sparse graph is either (i) parallel (ii) series or (iii) prime, in which case it corresponds to a spider $G[M]$. In general, we might therefore additionally distinguish prime modules M as those where $G[M]$ is a spider (called *spider modules*) and those where $G[M]$ is not a spider (which we still call simply *prime modules*).

3.5 Useful Properties of Modular Partitions

First, we briefly summarize the relationship between cographs G and the modular decomposition $\text{MD}(G)$.

Theorem 3.5 ([CLSB81, BLS99]) *Let $G = (V, E)$ be an arbitrary graph. Then the following statements are equivalent.*

1. G is a cograph.
2. G does not contain induced paths on four vertices P_4 .
3. $\text{MDT}(G)$ is the cotree of G and hence, has no inner vertices labeled with “prime”.
4. Any non-trivial induced subgraph of G has at least one pair of twins $\{x, y\}$.

For later explicit reference, we summarize in the next theorem several results that we already implicitly referred to in the discussion above.

Theorem 3.6 ([GPP10, HP10, Möh85]) *The following statements are true for an arbitrary graph $G = (V, E)$:*

- (T1) *The maximal modular partition $\mathbb{P}_{\max}(G)$ and the modular decomposition $\text{MD}(G)$ of G are unique.*
- (T2) *Let \mathbb{P} be a modular partition of G . Then $\tilde{\mathbb{P}} \subset \mathbb{P}$ is a (non-trivial strong) module of G/\mathbb{P} if and only if $\bigcup_{M \in \tilde{\mathbb{P}}} M$ is a (non-trivial strong) module of G .*
- (T3) *Let M be a module of G and $\{a, b, c, d\}$ be four vertices inducing a P_4 in G , then $|M \cap \{a, b, c, d\}| \leq 1$ or $\{a, b, c, d\} \subseteq M$.*
- (T4) *For any connected graph G with \bar{G} being connected, the quotient $G/\mathbb{P}_{\max}(G)$ is a prime graph.*
- (T5) *Let \mathbb{P}_{\max} be the maximal modular partition of $G[M]$, where M denotes a prime module of G and $\mathbb{P}' \subsetneq \mathbb{P}_{\max}$ be a proper subset of \mathbb{P}_{\max} with $|\mathbb{P}'| > 1$. Then, $\bigcup_{M' \in \mathbb{P}'} M' \notin \mathbb{M}(G)$.*

Statements (T1) and (T4) are clear. Statement (T2) characterizes the (non-trivial strong) module of G in terms of (non-trivial strong) modules $\tilde{\mathbb{P}}$ of G/\mathbb{P} . Statement (T3) clarifies that each induced P_4 is either entirely contained in a module or intersects a module in at most one vertex. Statement (T5) explains that none of the unions of elements of a maximal modular partition of $G[M]$ are modules of G . Hence, only the prime module M itself and, by Lemma 3.1, the elements $M' \in \mathbb{P}_{\max}$ are modules of G .

4 Cograph Editing

4.1 Optimal Modul-Preserving Edit Sets

Given an arbitrary graph we are interested in the following optimization problem.

Problem 4.1 (Cograph Editing) *Given a graph $G = (V, E)$. Find a set $F \subseteq \binom{V}{2}$ of minimum cardinality s.t. $G^* = (V, E \triangle F)$ is a cograph.*

We will simply call an edit sets of minimum cardinality an *optimal edit set*.

The (decision version of the) cograph-editing problem is NP-complete [LWGC11, LWGC12]. Nevertheless, the cograph-editing problem is fixed-parameter tractable (FPT) [PDdSS09]. Hence, for the parametrized version of this problem, i.e., for a given graph $G = (V, E)$ and a parameter $k \geq 0$ find a set F of at most k edges and non-edges so that $G^* = (V, E \triangle F)$ is a cograph, there is an algorithm with running time $O(6^k)$ [Cai96]. This FPT approach was improved in [LWGC11, LWGC12] to an $O(4.612^k + |V|^{4.5})$ time algorithm. These results are of little use for practical applications, because the constant k can become quite large. However, they provide deep insights into the structure of the class of P_4 -sparse graphs that slightly generalizes cographs.

In particular, the cograph-editing problem can be solved in polynomial time whenever the input graph is P_4 -sparse [LWGC11, LWGC12]. The key observation is that every strong *prime* module M of a P_4 -sparse graph G is a spider module. The authors then proceed to show that it suffices to edit a fixed number of (non)legs, i.e., only (non)edges xy with $x \in K$ and $y \in S$ in $G[M]$, for all such spider modules to eventually obtain an optimally edited cograph. The resulting algorithm to optimally edit a P_4 -sparse graph to a cograph, EDP4, runs in $O(|V| + |E|)$ -time.

In the following will frequently make use of a result by Guillemot *et al.* [GPP10] that is based on the following

Lemma 4.2 ([GPP10]) *Let $G = (V, E)$ be an arbitrary graph and let M be a non-trivial module of G . If F_M is an optimal edge-edition set of the induced subgraph $G[M]$ and F_{opt} is an optimal edge-edition set of G , then (i) $F = (F_{\text{opt}} \setminus F_{\text{opt}}[M]) \cup F_M$ is an optimal edge-edition set of G and (ii) $F_{\text{opt}}[M]$ contains all (non-)edges $xy \in F_{\text{opt}}$ with $x, y \in M$.*

Proposition 4.3 ([GPP10]) *Every graph $G(V, E)$ has an optimal edit set F_{opt} such that every module M of G is module of the cograph $G_{\text{opt}} = (V, E \triangle F_{\text{opt}})$.*

An edit set as described in Prop. 4.3 is called *module-preserving*. Their importance lies in the fact that module-preserving edit sets update either all or none of the edges between any two disjoint modules.

In the following Remark we collect a few simple consequences of our considerations so far.

Remark 4.4 *By Theorem 3.6 (T3) and definition of cographs, all induced P_4 's of a graph are entirely contained in the prime modules. By Lemma 3.1, the maximal modular partition \mathbb{P}_{max} of $G[M]$ is a subset of the strong modules $\text{MD}(G) \subseteq \mathbb{M}(G)$, for all strong modules $M \in \text{MD}(G)$. Taken together with Lemma 4.2, Proposition 4.3 and Theorem 3.6 (T1), this implies that it suffices to solve the cograph-editing problem for G independently on each of G 's strong prime modules [GPP10].*

An optimal module-preserving edit-set F_{opt} on G therefore induces optimal edit-sets F_M on $G[M]$ for any M , and thus also optimal edit-set $F_{\text{opt}}(M, \mathbb{P}_{\text{max}})$ on $G[M]/\mathbb{P}_{\text{max}}$, where $\mathbb{P}_{\text{max}} = \{M_1, \dots, M_k\}$ is again the maximal modular partition of $G[M]$ and M is a module of $\mathbb{M}(G)$. The edit set $F(M, \mathbb{P}_{\text{max}})$ has the

following explicit representation:

$$F(M, \mathbb{P}_{\max}) := \{\{M_i, M_j\} \mid M_i, M_j \in \mathbb{P}_{\max} \exists x \in M_i, y \in M_j \text{ with } \{x, y\} \in F_{\text{opt}}[M]\}.$$

4.2 Optimal Module Merge Deletes All P_4 's

Since cographs are characterized by the absence of induced P_4 's, we can interpret every cograph-editing method as the removal of all P_4 's in the input graph with a minimum number of edits. A natural strategy is therefore to detect P_4 's and then to decide which ones must be edited. Optimal edit sets are not necessarily unique. A further difficulty is that editing an edge of a P_4 can produce new P_4 's in the updated graph. Hence we cannot expect *a priori* that local properties of G alone will allow us to identify optimal edits.

By Remark 4.4, on the other hand, it is sufficient to edit within the prime modules. We therefore focus on the maximal modular partition $\mathbb{P}_{\max} = \mathbb{P}_{\max}(G[M])$ of $G[M]$, where $M \in \text{MD}(G)$ is a strong prime module of G . Since $G[M]/\mathbb{P}_{\max}$ is prime, it does not contain any twins. Now suppose we have edited G to a cograph G_{opt} using an optimally module-preserving edit set. Then $G_{\text{opt}}[M]$ is a cograph and by Lemma 3.2 the quotient $G_{\text{opt}}[M]/\mathbb{P}_{\max}$ is also a cograph. Therefore, $G_{\text{opt}}[M]/\mathbb{P}_{\max}$ contains at least one pair of twins $\{M_i, M_j\}$, where M_i and M_j are, by construction, children of the prime module $M \in \text{MD}(G)$.

This consideration suggests that it might suffice to edit the out_{M_i} - and out_{M_j} -neighborhoods in G in such a way that M_i and M_j become twins in an optimally edited cograph G_{opt} . In the following we will show that this is indeed the case.

We first show that twins are “safe”, i.e., that we never have to edit edges within a subgraph $G[M]$ induced by an equivalence classes M of the twin relation \mathcal{T} .

Lemma 4.5 *Let $G = (V, E)$ be a non-cograph, F be an arbitrary cograph edit set s.t. $G' = (V, E \triangle F)$ is the resulting cograph and suppose that $G' \triangle e$ is a non-cograph for all $e \in F$. Then $\{x, y\} \notin F$ for twins x, y in G' .*

Proof: Since G' is a cograph it contains at least one pair of twins x, y . First assume that x and y are false twins and thus, $xy \notin E(G')$. Assume, for contradiction, that $xy \in F$. By assumption, $G' + xy$ is not a cograph and thus there is an induced P_4 containing the edge xy in $G' + xy$. All such P_4 's that contain the edge xy are (up to symmetries and isomorphism) of the form (i) $a - x - y - b$ or (ii) $x - y - b - a$. Since x and y are false twins in G' , we have $N_{G'}(x) = N_{G'}(y)$ and hence, there must be an edge $xb \in E(G')$ and therefore, $xb \in E(G' + xy)$. But this implies that $G' + xy$ is still cograph, the desired contradiction.

Now suppose that x and y are true twins, i.e., $xy \in E(G')$. Assume, for contradiction, that $xy \in F$ and thus $G' - xy$ is not a cograph. Hence there must be an induced P_4 containing x and y in $G' - xy$. All such P_4 's containing x and y are (up to symmetries and isomorphism) of the form (i) $x - a - y - b$ or (ii) $x - a - b - y$. Since x and y are true twins in G' , we have $N_{G'}(x) \setminus \{y\} = N_{G'}(y) \setminus \{x\}$. This implies that there is the edge $xb \in E(G' - xy)$ in both case (i) and (ii). Therefore $G' - xy$ is a cograph, a contradiction. □

Theorem 4.6 *Let $G = (V, E)$ be an arbitrary graph, F_{opt} be an optimal cograph edit set for G , $G_{\text{opt}} = (V, E \triangle F_{\text{opt}})$ the resulting cograph and \mathcal{T} be the twin relation on G_{opt} . Then for each equivalence class $M \sqsubseteq \mathcal{T}$ it holds that:*

- (i) M is a module of G_{opt} that does not contain any other non-trivial strong module of $\mathbb{M}(G_{\text{opt}})$.
- (ii) $G_{\text{opt}}[M]$ induces either an independent set $\overline{K_{|M|}}$ or a complete graph $K_{|M|}$.
- (iii) For all $x, y \in M$ it holds that $\{x, y\} \notin F_{\text{opt}}$ and thus, $G[M] \simeq G_{\text{opt}}[M]$.

Proof: Statements (i) and (ii) are an immediate consequences of Proposition 3.3. If $e \in F_{\text{opt}}$ then $G_{\text{opt}} \triangle e$ is a non-cograph; otherwise $F_{\text{opt}} \setminus \{e\}$ would be an edit set with smaller cardinality, contradicting the optimality of F_{opt} . Thus we can apply Lemma 4.5 to infer statement (iii). \square

Corollary 4.7 Let $G = (V, E)$ be a prime graph, F_{opt} be an optimal cograph edit set, $G_{\text{opt}} = (V, E \triangle F_{\text{opt}})$ the resulting cograph and \mathcal{T} be the twin relation on G_{opt} . Then $V \not\models \mathcal{T}$.

Proof: As $G = (V, E)$ is a prime graph we know that $F_{\text{opt}} \neq \emptyset$. If $V \models T$, then Theorem 4.6 implies that $\{x, y\} \notin F_{\text{opt}}$ for all $x, y \in V$ and hence, $F_{\text{opt}} = \emptyset$, a contradiction. \square

The following definitions are important for the concepts for the “module merge process” that we will extensively use in our approach.

Definition 2 (Module Merge) Let G and H be arbitrary graphs on the same vertex set V with their corresponding sets of all modules $\mathbb{M}(G)$ and $\mathbb{M}(H)$, resp. We say that a subset $M' = \{M_1, \dots, M_k\} \subseteq \mathbb{M}(G)$ of modules is merged (w.r.t. H) – or, equivalently, the modules in M' are merged (w.r.t. H) – if (a) each of the modules in M' is a module of H , and (b) the union of all modules in M' is a module of H but not of G . More formally, the modules in M' are merged (w.r.t. H), if

- (i) $M_1, \dots, M_k \in \mathbb{M}(H)$,
- (ii) $M = \cup_{i=1}^k M_i \in \mathbb{M}(H)$, and
- (iii) $M \notin \mathbb{M}(G)$.

If $\{M_1, \dots, M_k\} \subseteq \mathbb{M}(G)$ is merged to a new module $M \in \mathbb{M}(H)$ we will write this as $M_1 \boxplus \dots \boxplus M_k = \boxplus_{i=1}^k M_i \rightarrow M$.

When modules M_1, \dots, M_k of G are merged w.r.t. H then all vertices in $M = \cup_{i=1}^k M_i$ must have the same out_M -neighbors in H , while at least two vertices $x \in M_i, y \in M_j, 1 \leq i \neq j \leq k$ must have different out_M -neighbors in G .

Definition 3 (Module Merge Edit) Let $G = (V, E)$ be an arbitrary graph and F be an arbitrary edit set resulting in the graph $H = (V, E \triangle F)$. Assume that $M_1, \dots, M_k \in \mathbb{M}(G)$ are modules that have been merged w.r.t. H resulting in the module $M = \cup_{i=1}^k M_i \in \mathbb{M}(H)$. Then

$$F_H(\boxplus_{i=1}^k M_i \rightarrow M) = \{(x, v) \in F \mid x \in M, v \notin M\} \quad (1)$$

The edit set $F_H(\boxplus_{i=1}^k M_i \rightarrow M)$ comprises exactly those (non)edges of F that have been edited so that all vertices in M have the same out_M -neighborhood in H . In particular, it contains only (non)edge of F that are not entirely contained in $G[M]$.

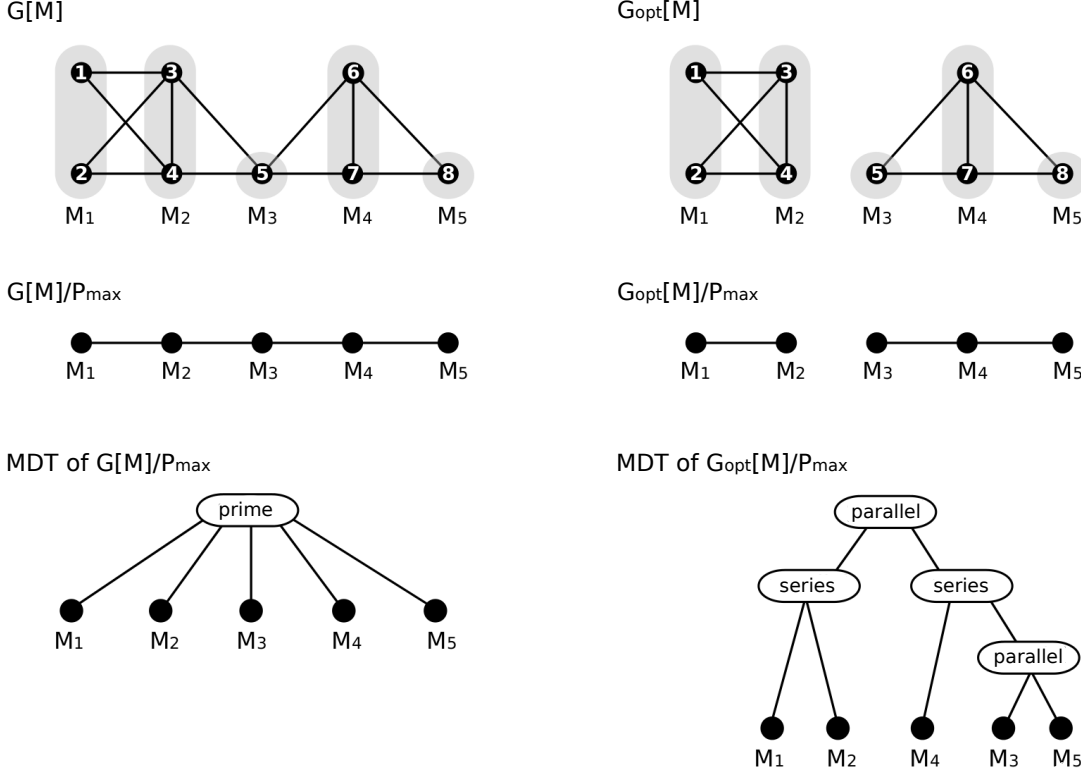


Fig. 1: Assume we are given a non-cograph G that contains a strong prime modules $M \in \text{MD}(G)$ so that $G[M]$ is the graph in the upper left part of this picture. Moreover, assume there is an optimal module-preserving edit set F_{opt} transforming G to a cograph G_{opt} so that $\{3, 5\}, \{4, 5\} \in F_{\text{opt}}$. Hence, $G_{\text{opt}}[M]$ is the graph in the upper right part. Let $\mathbb{P}_{\max} = \mathbb{P}_{\max}(G[M]) = \{M_1, \dots, M_5\}$ be the maximal modular partition of $G[M]$. The modules M_1, \dots, M_5 are highlighted as gray parts in $G[M]$ and $G_{\text{opt}}[M]$. Now, $G[M]/\mathbb{P}_{\max}$ is prime and contains no twins, while $G_{\text{opt}}[M]/\mathbb{P}_{\max}$ is a cograph and contains therefore twins. The twin relation \mathcal{T} on $G_{\text{opt}}[M]/\mathbb{P}_{\max}$ has equivalence classes $N = \{M_1, M_2\}$, $N' = \{M_4\}$ and $N'' = \{M_3, M_5\}$. Hence, the modules M_1 and M_2 , as well as, M_3 and M_5 are merged w.r.t. $G_{\text{opt}}[M]/\mathbb{P}_{\max}$. Therefore, the modules $N_V = M_1 \cup M_2 = \{1, 2, 3, 4\}$ and $N'_V = M_3 \cup M_5 = \{5, 8\}$ have been obtained by merging modules of $G[M]$ w.r.t. $G_{\text{opt}}[M]$ and in particular, w.r.t. G_{opt} . In symbols, $M_1 \sqcup M_2 \rightarrow N_V$ and $M_3 \sqcup M_5 \rightarrow N'_V$.

Therefore, instead of focussing on algorithms that optimally edit induced P_4 's one can equivalently ask for optimal edit sets that resolve such prime modules M , that is, one asks for the minimum number of edits that adjust the neighborhoods of modules that are children of M in $\text{MDT}(G)$ so that these modules become twins until the module M becomes a non-prime module, see Theorem 4.10.

Lemma 4.8 *Let M be a strong prime module of a given graph $G = (V, E)$ and \mathbb{P}_{\max} be the maximal modular partition of $G[M]$. Moreover, let F be an arbitrary edit set resulting in the graph $H = (V, E \triangle F)$. Assume that $\mathbb{M} = \{M_1, \dots, M_k\} \subseteq \mathbb{P}_{\max} \subseteq \mathbb{M}(G)$ is a set of modules that are merged w.r.t. H .*

Then for any distinct indices $a, b, c \in \{1, \dots, k\}$ it holds that $M_a \sqcup M_b \rightarrow M_{ab}$, $M_b \sqcup M_a \rightarrow M_{ba}$, and $M_{ab} = M_{ba}$ in H . Moreover, $M_a \sqcup (M_b \sqcup M_c) \rightarrow M_{a(bc)}$ and $(M_a \sqcup M_b) \sqcup M_c \rightarrow M_{(ab)c}$ satisfy $M_{a(bc)} = M_{(ab)c}$, i.e., the merge operation is associative and commutative. The merging of any subset \mathbb{M} of modules w.r.t. a graph H therefore is well-defined and independent of the individual merging steps.

Furthermore,

$$\begin{aligned} F(M_a \sqcup M_b \sqcup M_c \rightarrow M_{abc}) &= F(M_a \sqcup M_b \rightarrow M_{ab}) \\ &\cup (F(M_{ab} \sqcup M_c \rightarrow M_{abc}) \setminus F(M_a \sqcup M_b \rightarrow M_{ab})) \end{aligned}$$

with $M_a \sqcup M_b \rightarrow M_{ab}$.

Proof: For commutativity, we show first that $M_a \sqcup M_b \rightarrow M_{ab}$ for every pair of distinct $a, b \in \{1, \dots, k\}$ By definition of “ \sqcup ” we have $M_a, M_b \in \mathbb{M}(H)$, and thus, Condition (i) of Def. 2 is satisfied. Moreover, Thm. 3.6 (T5) implies that $M_a \cup M_b \notin \mathbb{M}(G)$ and hence, Condition (iii) of Def. 2 is satisfied. For Condition (ii) we have to show that $M_a \cup M_b = M_{ab} \in \mathbb{M}(H)$. Assume for contradiction that $M_a \cup M_b \notin \mathbb{M}(H)$, then there is a vertex $v \in V$ so that v is in the out_{M_a} -neighborhood, but not in the out_{M_b} -neighborhood w.r.t. H , or *vice versa*. However, this remains true if we consider $\bigcup_{i=1}^k M_i$, which implies that $\bigcup_{i=1}^k M_i \notin \mathbb{M}(H)$, and hence $\mathbb{M} = \{M_1, \dots, M_k\}$ is not merged w.r.t. H , a contradiction. Finally, $M_a \sqcup M_b \rightarrow M_{ab}$ implies that $M_{ab} = M_a \cup M_b = M_b \cup M_a = M_{ba}$, and thus \sqcup is commutative.

By similar arguments one shows that Condition (i), (ii), and (iii) of Def. 2 are satisfied for $M_a \sqcup M_b \sqcup M_c$. Hence, since $M_a \sqcup M_b \sqcup M_c \rightarrow M_{abc}$ implies that $M_{abc} = M_a \cup M_b \cup M_c$, associativity of \sqcup follows again directly from the associativity of the set union.

To see that the last property for F is satisfied, note that

$$\begin{aligned} &F(M_a \sqcup M_b \sqcup M_c \rightarrow M_{abc}) \\ &= \{(x, v) \in F \mid x \in M_{abc}, v \notin M_{abc}\} \\ &= \{(x, v) \in F \mid x \in M_{ab}, v \notin M_{ab}\} \cup \{(x, v) \in F \mid x \in M_{abc} \setminus M_{ab}, v \notin M_{abc} \setminus M_{ab}\} \\ &= F(M_a \sqcup M_b \rightarrow M_{ab}) \cup (F(M_{ab} \sqcup M_c \rightarrow M_{abc}) \setminus F(M_a \sqcup M_b \rightarrow M_{ab})) \end{aligned}$$

□

Let G be an arbitrary graph and F_{opt} be a minimum cardinality set of edits that applied to G result in the cograph G_{opt} . We will show that every module-preserving edit set F_{opt} can be expressed completely by means of module merge edits. To this end, we will consider the strong prime modules $M \in \text{MD}(G)$ of the given graph G (in particular certain submodules of M that do not share the same out-neighborhood) and adjust their out-neighbors to obtain new modules as long as M stays a prime module. This procedure is repeated for all prime modules of G , until no prime modules are left in G .

As mentioned above, if G is not a cograph there must be a strong prime module M in G . Let \mathbb{P}_{\max} be the maximal modular partition of $G[M]$. Theorem 3.6 implies that $G[M]/\mathbb{P}_{\max}$ is prime and thus, does not contain twins. However, if F_{opt} is module preserving, then Lemma 3.2 implies that the graph $G_{\text{opt}}[M]/\mathbb{P}_{\max}$ is a cograph and thus, contains non-trivial twins by Theorem 4.

Hence, we aim at finding particular submodules $M_1, M_2, \dots, M_k \subset M$ in G that have to be merged so that they become twins in $G_{\text{opt}}[M]/\mathbb{P}_{\text{max}}$. As the following theorem shows, repeated application of this procedure to all prime modules of G will completely resolve those prime modules resulting in G_{opt} . An illustrative example is shown in Figure 1.

For the proof of the final Theorem 4.10 we first establish the following result.

Lemma 4.9 *Let $G = (V, E)$ be graph, F_{opt} be an optimal module-preserving cograph edit set, and $G_{\text{opt}} = (V, E \triangle F_{\text{opt}})$ be the resulting cograph. Furthermore, let M be an arbitrary strong prime module of G that does not contain any other strong prime module, and $\mathbb{P}_{\text{max}} := \mathbb{P}_{\text{max}}(G[M]) = \{M_1, \dots, M_k\}$ be the maximal modular partition of $G[M]$. Moreover, let $F' = \{\{x, y\} \in F_{\text{opt}} \mid \exists M_i, M_j \in F_{\text{opt}}(M, \mathbb{P}_{\text{max}}) \text{ with } x \in M_i, y \in M_j\}$, where $F_{\text{opt}}(M, \mathbb{P}_{\text{max}})$ is the edit set to implied by F_{opt} on $G[M]/\mathbb{P}_{\text{max}}$ defined in Remark 4.4.*

Then, every strong prime module of $H = (V, E \triangle F')$ is a strong prime module of G . Moreover, $\mathbb{P}_{\text{max}}(H[M']) = \mathbb{P}_{\text{max}}(G[M'])$ holds for each strong prime module M' of H .

Proof: First note, that there is no other module $M' \subsetneq M$ containing induced P_4 's because M does not contain any other strong prime module $M' \subsetneq M$ and because by Theorem 3.6 (T5) the union $\cup_{M' \in P'} M'$ is not a module of G for any of the subsets $P' \subsetneq \mathbb{P}_{\text{max}}$. Together with Lemma 4.2 this implies that $F' = \{\{x, y\} \in F_{\text{opt}} \mid x, y \in M\}$. Thus, $H[M]$ is a cograph as otherwise, there would be induced P_4 's contained in $H[M]$ and there are no further edits in $F_{\text{opt}} \setminus F'$ to remove these P_4 's. Thus, such P_4 's would remain in G_{opt} , a contradiction.

Let M' be an arbitrary strong prime module of H . Note, since F' does not affect the out_M -neighborhood, M is still a module of H . Since M' is a strong module in H we have, therefore, either, $M' \subseteq M$, $M \subsetneq M'$ or $M \cap M' = \emptyset$ in G' .

Assume that $M' \subseteq M$. Since M' is prime in H it follows that $H[M']$ is a non-cograph, a contradiction, since $H[M']$ is an induced subgraph of the cograph $H[M]$. Hence, the case $M' \subseteq M$ cannot occur. If $M \subsetneq M'$ or $M \cap M' = \emptyset$, then the $\text{out}_{M'}$ -neighborhood of any vertex contained in M' have not been affected by F' , and thus, M' is also a module of G .

It remains to show for the cases $M \subsetneq M'$ or $M \cap M' = \emptyset$ that the module M' of G is also strong and prime. If $M \cap M' = \emptyset$, then F' does not affect any vertex of M' and hence, M' must be a strong prime module of G and, in particular, $\mathbb{P}_{\text{max}}(H[M']) = \mathbb{P}_{\text{max}}(G[M'])$.

Let $M \subsetneq M'$ and assume for contradiction that M' is not strong in G . Thus, M' must overlap some other module M'' in G . However, since M is a strong module of G , M cannot overlap M'' and hence, $M \cap M'' = \emptyset$. However, as F' does only affect the vertices within M and, in particular, none of the vertices of M'' and since M' and M'' overlap in G , they must also overlap in H , a contradiction. Thus, M' is a strong module of G .

Furthermore, let again $M \subsetneq M'$ and assume that M' is not prime in G . Now, let \mathbb{P}'_{max} be the maximal modular partition of $G[M']$. Since M and M' are strong modules in G and $M \subsetneq M'$, we have either $M \in \mathbb{P}'_{\text{max}}$ or $M \subsetneq M'' \in \mathbb{P}'_{\text{max}}$ for some strong module $M'' \in \mathbb{P}_{\text{max}}$. Hence, all modules of \mathbb{P}_{max} are entirely contained in the modules of \mathbb{P}'_{max} . In particular, this implies that we have not changed the $\text{out}_{M''}$ -neighborhood for any $M'' \in \mathbb{P}'_{\text{max}}$, and therefore, the maximal modular partition of $G[M']$ is also the maximal modular partition of $H[M']$, i.e., $\mathbb{P}_{\text{max}}(H[M']) = \mathbb{P}_{\text{max}}(G[M'])$. However, if M' is not prime in G , then $G[M']/\mathbb{P}'_{\text{max}}$ is either totally disconnected (M' is parallel) or a complete graph (M' labeled series), while $H[M']/\mathbb{P}'_{\text{max}}$ is prime (hence, it does contain only trivial modules). Clearly, if $G[M']/\mathbb{P}'_{\text{max}}$ is totally disconnected or a complete graph, any subset $P' \subseteq P$ forms a module and thus, $G[M']/\mathbb{P}'_{\text{max}}$ does not contain only

trivial modules. In summary, $G[M']/\mathbb{P}'_{\max} \not\cong H[M']/\mathbb{P}'_{\max}$, a contradiction, since we have not changed the $\text{out}_{M''}$ -neighborhood for any $M'' \in \mathbb{P}'_{\max}$. \square

Theorem 4.10 *Let $G = (V, E)$ be graph, F_{opt} be an optimal module-preserving cograph edit set, and $G_{\text{opt}} = (V, E \triangle F_{\text{opt}})$ be the resulting cograph. Denote by $\text{MDP}(G) \subseteq \text{MD}(G)$ the set of strong prime modules of G .*

Furthermore, let \mathbb{M}_i denote a set of modules that have been merged w.r.t. G_{opt} resulting in the new module $\mathcal{M}_i = \bigcup_{M \in \mathbb{M}_i} M \in \mathbb{M}(G_{\text{opt}})$, where \mathbb{M}_i is a subset of some $\mathbb{P}_{\max}(G[M'])$ for some $M' \in \text{MDP}(G)$. In other words, each \mathbb{M}_i contains only the children of strong prime modules of G .

Let $\mathbb{M}_1, \dots, \mathbb{M}_r$ be all these sets of modules of G that have been merged w.r.t. G_{opt} into respective modules $\mathcal{M}_1, \dots, \mathcal{M}_r$ and $\mathcal{M} = \{\mathcal{M}_1, \dots, \mathcal{M}_r\}$ denote the set of these resulting new modules in G_{opt} . Then

$$F_{\text{opt}} = F_{\mathcal{M}} := \bigcup_{i=1}^r F_{G_{\text{opt}}} \left(\bigsqcup_{M \in \mathbb{M}_i} M \rightarrow \mathcal{M}_i \right)$$

Proof: The proof follows an iterative process, starting with the input graph $G^0 = G$, and proceeds by stepwisely editing *within* certain strong prime modules M resulting in new graphs $G^1, G^2, \dots, G^n = G_{\text{opt}}$ and $F_{\text{opt}} = F_{\mathcal{M}}$ for some integer n . In each step, which leads from a non-cograph G^k to G^{k+1} , we operate on one strong prime module $M \in \text{MD}(G^k)$ of G^k that does not contain any other strong prime module. We write $\mathbb{P}_{\max}^k := \mathbb{P}_{\max}(G^k[M])$ for the maximal modular partition of $G^k[M]$. Theorem 3.6 (T4) implies that $G^k[M]/\mathbb{P}_{\max}^k$ is prime and thus does not contain twins. Lemma 3.2 implies that $G_{\text{opt}}[M]/\mathbb{P}_{\max}^k$ is a cograph and hence, by Theorem 3.5 contains twins.

We will show in *PART 1* that each equivalence class N of the twin relation $N \sqsubseteq \mathcal{T}$ on $G_{\text{opt}}[M]/\mathbb{P}_{\max}^k$ yields a set of vertices $N_V \subseteq V$, where each N_V is a module of G_{opt} but not of G^k . In particular, every such N_V is obtained by merging modules of G^k only so that only N_V is affected. Application of merge edit operations contained in F_{opt} to obtain these new modules N_V results in the new graph G^{k+1} . The procedure is then repeated unless the new graph G^{k+1} equals G_{opt} .

We then show in *PART 2*, that none of the new modules N_V is a module of the starting graph G . Furthermore we will see that N_V is obtained not only by merging modules of G^k but also by merging modules of G . Finally we use these two results to show that $F_{\text{opt}} = F_{\mathcal{M}}$.

PART 1:

We start with the base case, and show how to obtain the graph G^1 from $G^0 := G$. Set $F^0 = F_{\text{opt}}$. W.l.o.g., assume that G^0 is not a cograph and thus that it contains prime modules. Let $M \in \text{MD}(G^0)$ be a strong prime module of G^0 that does not contain any other strong prime module.

By the preceding arguments, $G^0[M]/\mathbb{P}_{\max}^0$ is prime and thus does not contain twins, while $G_{\text{opt}}[M]/\mathbb{P}_{\max}^0$ is a cograph and contains twins. In particular, there must be vertices (representing strong modules of G^0) in $G^0[M]/\mathbb{P}_{\max}^0$ that become twins in $G_{\text{opt}}[M]/\mathbb{P}_{\max}^0$. Let \mathcal{T} be the twin relation of $G_{\text{opt}}[M]/\mathbb{P}_{\max}^0$ and denote, for a given equivalence classes $N \sqsubseteq \mathcal{T}$, by $N_V \subseteq V$ the set of twins contained in N , i.e., $N_V := \bigcup_{M_j \in N} M_j = \bigcup_{M_j \in N} \{v \in M_j\}$. By Theorem 4.6, each $N \sqsubseteq \mathcal{T}$ is a module of $G_{\text{opt}}[M]/\mathbb{P}_{\max}^0$. Lemma 3.2 implies that \mathbb{P}_{\max}^0 is a modular partition of G_{opt} and thus we can apply Theorem 3.6 (T2) to conclude that N_V is a module of $G_{\text{opt}}[M]$. Since F_{opt} is module preserving, it follows that M is a module of G_{opt} and thus, by Lemma 3.1, N_V is a module of G_{opt} . Corollary 4.7 furthermore implies that $N_V \neq M$

for all $N \sqsubseteq \mathcal{T}$. Moreover, by Theorem 3.6 (T5), none of the equivalence classes $N \sqsubseteq T$ with $|N| > 1$ are modules of $G^0[M]/\mathbb{P}_{\max}^0$. Hence, by Theorem 3.6 (T2), N_V is not a module of $G^0[M]$ for all $N \sqsubseteq \mathcal{T}$ with $|N| > 1$ and hence, Lemma 3.1 implies that N_V is not a module of $G^0 = G$, and for the respective vertex set N_V we have both $N_V \notin \mathbb{M}(G^0)$ and $N_V \in \mathbb{M}(G_{\text{opt}})$, the modules contained in N are merged to N_V w.r.t. G_{opt} , in symbols $\sqcup_{M_i \in N} M_i \rightarrow N_V$ for all $N \sqsubseteq \mathcal{T}$ with $|N| > 1$.

We write $\mathcal{M}^1 := \{N_V \mid N_V = \sqcup_{M_i \in N} M_i, N \sqsubseteq \mathcal{T}, |N| > 1\}$ for the set of all new modules N_V that are the result of merging modules of $G^0[M]$. We continue to construct the graph G^1 . To this end, we set $F^0 = F_{\text{opt}}$ and show that the subset $E^0 \subseteq F_{\text{opt}}$, where E^0 contains all pairs of vertices $\{x, y\}$ with $x \in N_V$ and $y \in M \setminus N_V$ for all $N_V \in \mathcal{M}^1$, is non-empty. Let $F(\mathcal{T}) = \{(M_i, M_j) \in F^0(M, \mathbb{P}_{\max}^0) \mid M_i \in N, M_j \notin N, N \sqsubseteq \mathcal{T}, |N| > 1\}$, where $F^0(M, \mathbb{P}_{\max}^0)$ is the edit set implied by F^0 as defined in Remark 4.4.

Theorem 4.6 implies that for all $x, y \in N_V$ it holds that $\{x, y\} \notin F_{\text{opt}}$ for any $N \sqsubseteq \mathcal{T}$. As either all or none of the edges between any two modules are edited, we conclude that $\{M_i, M_j\} \notin F(\mathcal{T})$ holds for all $N \sqsubseteq \mathcal{T}$ and all $M_i, M_j \in N$. However, since $G^0[M]/\mathbb{P}_{\max}^0$ does not contain any pair of twins, while $G_{\text{opt}}[M]/\mathbb{P}_{\max}^0$ does, it must have $F(\mathcal{T}) \neq \emptyset$. Hence, $E^0 = \{\{x, y\} \in F^0 \mid x \in M_i, y \in M_j, (M_i, M_j) \in F(\mathcal{T})\} \neq \emptyset$. Theorem 4.6 implies $\{x, y\} \notin E^0$ for all $x \in M_i, y \in M_j$, all $M_i, M_j \in N$ and all $N \sqsubseteq \mathcal{T}$. Thus, $\{x, y\} \notin E^0$ for all $x, y \in N_V$. Since $E^0 \neq \emptyset$, it contains only pairs of vertices $\{x, z\}$ with $x \in N_V \subseteq M$ and $z \in M \setminus N_V$. In other words, E^0 comprises only pairs of vertices $\{x, y\} \subset M$ that have been edited to merge the modules of $G^0[M]$ resulting in the new modules $N_V \in \mathcal{M}^1$. Note that there might be additional edits $\{x, y\} \in F_{\text{opt}} \setminus E^0$ for some $x \in N_V$ and $y \in V \setminus N_V$. Nevertheless we have $E^0 \subseteq \bigcup_{N \sqsubseteq \mathcal{T}} F_{G_{\text{opt}}}(\sqcup_{M_i \in N} M_i \rightarrow N_V) \subseteq F_{\mathcal{M}}$.

Finally, set $G^1 = (V, E \triangle E^0)$. Thus, all $N_V \in \mathcal{M}^1$ are modules of G^1 . If $G^1 = G_{\text{opt}}$, then $E^0 = F_{\text{opt}}$. By construction $E^0 \subseteq F_{\mathcal{M}} \subseteq F_{\text{opt}}$. Hence we can conclude that $F_{\mathcal{M}} = F_{\text{opt}}$ and we are done.

Now we turn to the general editing step. If G^k , $k \geq 1$ is not a cograph, then we define the set $F^k = F^{k-1} \setminus E^{k-1}$. We can re-use exactly the same arguments as above. Starting with a strong prime module $M \in \text{MD}(G^k)$ (that does not contain any other strong prime module), we show that the resulting set E^k is not empty and comprises (non)edges in $F^k \subseteq F_{\text{opt}}$ so that $E^k \subseteq F_{\mathcal{M}}$. In particular, we find that E^k contains only the (non)edges that have been edited so that for all new modules $N_V \in \mathcal{M}^{k+1}$ of the graph $G^{k+1} = (V, E \triangle (\bigcup_{j=0}^k E^j))$ it holds that $N_V \notin \mathbb{M}(G^k)$, $N_V \in \mathbb{M}(G_{\text{opt}})$ and that N_V is the union of modules contained in $\mathbb{M}(G^k)$ that are, in particular, children of the chosen prime module in G^k .

PART 2:

It remains to show that every $N_V \in \mathcal{M}^{k+1}$ has the following two properties:

- (a) $N_V \notin \mathbb{M}(G)$; and
- (b) N_V is a module that has been obtained by merging modules that are children of prime modules of G and not only by merging such modules of G^k .

We first prove the following

Claim 1: Any module of $G^0 = G$ is also a module of G^k .

Proof of Claim 1.

We proceed by induction. Let $k = 1$ and thus, $G^1 = (V, E \triangle E^0)$. Let M^* be a module of $G^0 = G$. If $\{x, a\} \notin E^0$ for all vertices $x \in M^*$ then M^* is a module in G^1 because the out_{M^*} -neighborhood in G^1 is

not changed. Now assume that there is some vertex $x \in M^*$ with $\{x, a\} \in E^0$. If all such a with $\{x, a\} \in E^0$ are also contained in M^* , then M^* is still a module in G^1 . Therefore, in what follows assume that $a \notin M^*$.

Let M be the strong prime module that has been used to edit G^0 to G^1 in our construction above. Since for all $\{x, a\} \in E^0$ we have by construction $x, a \in M$, we can conclude that $M \cap M^* \neq \emptyset$. Since M is a strong module we have $M \subseteq M^*$ or $M^* \subseteq M$. However, from $a \in M$ and $a \notin M^*$ it follows that $M^* \subsetneq M$. Write \mathbb{P}_{\max}^0 for the maximal modular partition of $G^0[M]$. Theorem 3.6 (T5) implies that for any non-trivial subset \mathbb{P}' of \mathbb{P}_{\max}^0 the union of elements cannot be a module of G^0 . In other words, M^* is not the union of elements of any such subset \mathbb{P}' . Therefore, if $M^* \subsetneq M$, then $M^* = M_i$ for some module $M_i \in \mathbb{P}_{\max}^0$. If $M_i = \{x\}$, then M_i trivially remains a module of G^1 . Hence assume $|M_i| > 1$. By construction $M_i \subseteq N_V$ for some $M_i \in N \sqsubseteq \mathcal{T}$. There are two cases: (1) $|N| > 1$ and thus $M_i \subsetneq N_V$, and (2) $|N| = 1$, i.e., $N = \{M_i\}$, and thus $N_V = M_i$ is not merged.

Case (1) Assume first that the edge $(x, a) \in E^0$ with $x \in M_i$ and $a \notin M_i$ was added. Since F_{opt} is module preserving, $N_V \in \mathbb{M}(G_{\text{opt}})$. Hence, by construction of E^0 , we can conclude that all vertices $y \in N_V \subseteq M$ must be adjacent to a in $G^1[M]$. Since $M_i \subseteq N_V$, all $y \in M_i$ must be adjacent to a in $G^1[M]$. Analogously, if the edge (x, a) has been removed, then all $y \in M_i$ must be non-adjacent to a . Hence, elements in M_i have the same out_{M_i} -neighborhood in $G^1[M]$ and thus, M_i is a module of $G^1[M]$. Since our construction does not change the out_M -neighborhood, i.e., M is a module of G^1 , Lemma 3.1 implies that M_i is also a module of G^1 .

Case (2) Recall first that E^0 comprises only pairs of vertices $\{u, v\} \subset M$ that have been edited to merge the modules of $G^0[M]$ resulting in the new modules in \mathcal{M}^1 . Since N_V is not merged we have $N_V \notin \mathcal{M}^1$ and thus, we cannot directly assume that $\{y, a\}$ with $y \in N_V = M_i$ are in E^0 . However, as there was some edit $\{x, a\} \in E^0$ there must be some other merged module $N'_V \in \mathcal{M}^1$ with $a \in N'_V$. Moreover, since F_{opt} is module preserving it follows that M_i is a module of G_{opt} . Hence, if $\{x, a\} \in E^0 \subseteq F_{\text{opt}}$ was added, then all pairs of vertices $\{y, a\}$ with $y \in M_i$ not adjacent to a must be contained in F_{opt} and thus, in particular, $a \in N'_V$, which implies $\{y, a\}$ in E^0 . Analogous arguments apply if $\{x, a\}$ was removed. Therefore, either all vertices in M_i are adjacent to a or all of them as nonadjacent to a in $G^1[M]$. Therefore M_i is a module of $G^1[M]$. As in case (1) we can now argue that M_i is also a module of G^1 .

To summarize, all modules of $G^0 = G$ are modules of G^1 . Assume the statement is true for k and assume that G^k is not a cograph (since there is nothing more to show if G^k is a cograph). Applying the same arguments as above, we can infer that every module of G^k is also a module of G^{k+1} . Since all modules of G are by assumption also modules of G^k , they are also modules of G^{k+1} . \circ

Proof of Statement (a).

By Claim 1., every module of G is a module of G^k . Thus, if there is a subset $M \subseteq V$ that is not a module of G^k , then M is not a module of G . Since we have already shown that all modules $N_V \in \mathcal{M}^{k+1}$ of G^{k+1} are not modules of G^k , we conclude that none of the modules $N_V \in \mathcal{M}^{k+1}$ can be a module of G . This implies statement (a). \diamond

Proof of Statement (b).

By Lemma 4.9 and construction of G^k , we find that all strong prime modules of G^k are also strong prime modules of G^{k-1} . Therefore, by induction, any strong prime module of G^k is also a strong prime module of G .

It remains to show that the children of the chosen strong prime module M in G^k are also the children of M in G . In other words, we must show that for the maximal modular partitions we have that $\mathbb{P}_{\max}(G[M]) = \mathbb{P}_{\max}(G^k[M])$ for each $k \geq 1$.

We again proceed by induction: By construction, if M is the chosen strong prime module of G^0 that does not contain any other strong prime module in G^0 , then only children of M in G^0 are merged to obtain G^1 . Therefore, let M be the chosen strong prime module of G^1 that does not contain any other strong prime module in G^1 and that is used to obtain the graph G^2 . Lemma 4.9 implies that M is a strong prime module of G and $\mathbb{P}_{\max}(G[M]) = \mathbb{P}_{\max}(G^0[M]) = \mathbb{P}_{\max}(G^1[M])$. Thus, only children of prime modules of G have been merged to obtain the graph G^2 . Assume the statement is true for k . By analogous arguments as in the step from G^0 to G^1 , we can show that for the chosen strong prime module M in G^k that does not contain any other strong prime of G^k to obtain G^{k+1} , we have $\mathbb{P}_{\max}(G^{k-1}[M_k]) = \mathbb{P}_{\max}(G^k[M])$. Thus, only children of prime modules of G^{k-1} have been merged to obtain the graph G^{k+1} . However, since each strong prime module of G^k is a strong prime module of G and since by induction hypothesis $\mathbb{P}_{\max}(G^{k-1}[M]) = \mathbb{P}_{\max}(G[M])$ we obtain that $\mathbb{P}_{\max}(G^k[M]) = \mathbb{P}_{\max}(G[M])$, and thus only children of prime modules of G have been merged to obtain the graph G^{k+1} . \diamond

Finally, recall that $E^k \subseteq \{\{x, y\} \in F_{\text{opt}} \mid x \in N_V, y \in V \setminus N_V, N_V \in \mathcal{M}^k\} = \bigcup_{N \in \mathcal{T}} F_{G_{\text{opt}}}(\boxplus_{M_i \in N} M_i \rightarrow N_V) \subseteq F_{\mathcal{M}}$, where \mathcal{T} is the twin relation applied to $G_{\text{opt}}[M]/\mathbb{P}_{\max}^k$ of the chosen strong prime module M in G^k that does not contain any other strong prime module. By construction and because any module contained in $\mathcal{M}' = \{N_V \in M^j \mid 1 \leq j \leq k\}$ is obtained by merging the children of strong prime modules G only and since the children of strong prime modules are in particular modules of G , we have $\bigcup_{j=1}^k E^j \subseteq \bigcup_{j=1}^k \bigcup_{N_V \in \mathcal{M}^j} F_{G_{\text{opt}}}(\boxplus_{M \in N} M \rightarrow N_V) \subseteq F_{\mathcal{M}}$, where N comprises all modules that have been merged to obtain the respective module N_V contained in \mathcal{M}' .

This iterative process may lead to the situation that $\mathcal{M}' \subsetneq \mathcal{M}$, where \mathcal{M} denotes the set of all modules of G_{opt} that have been obtained by merging modules of G . However, since $E^k \subseteq F_{\text{opt}}$ and, in particular, $E^k \neq \emptyset$ if and only if G^k is a non-cograph (and thus contains prime modules), and because $|F^{k+1}| < |F^k|$, the iteration necessarily terminates for some finite n with $F^n = \emptyset$ and thus $G^n = (V, E \triangle (\bigcup_{j=1}^n E^j)) = G_{\text{opt}}$ and $\bigcup_{i=1}^n E^i = F_{\text{opt}}$. Since $\bigcup_{i=1}^n E^i \subseteq F_{\mathcal{M}'} \subseteq F_{\mathcal{M}} \subseteq F_{\text{opt}}$ we can finally conclude that $F_{\mathcal{M}} = F_{\text{opt}}$. \square

Theorem 4.10 implies that every module-preserving optimal edit set F_{opt} (which always exist) can be expressed as optimal module merge edits. Hence, instead of editing the induced P_4 's of a given graph G directly, one can *equivalently* resolve the strong prime modules by merging their children until no further prime module is left. At the first glance this result seems to be only of theoretical interest since the construction of optimal module merge edits is not easier than solving the editing problem.

The proof is constructive, however, and implies an alternative *exact* algorithm to solve the cograph editing problem, this time based on the stepwise resolution of the prime modules. Of course it has exponential runtime because in each step one needs to determine which of the modules have to be merged and which of the (non)edges have to be edited to obtain new modules. In particular, there are 2^n possible subsets for a prime module with n children in $\text{MDT}(G)$ that give all rise to modules that can be merged to new ones. Moreover, for each subset of modules that will be merged, there are exponentially many possibilities in the number of vertices to add or remove edges.

5 A modular-decomposition-based Heuristic for Cograph Editing

The practical virtue of our result is that it suggests an alternative strategy to construct heuristic algorithms for cograph editing. Our starting point is Lemma 4.8, which implies that we can construct F_{opt} by pairwise merging of children of prime modules.

Assume there is strong prime module M and \mathbb{M} denotes a subset children of M w.r.t. $\text{MDT}(G)$ that have been merged w.r.t. G_{opt} . Now, instead of merging all modules at once, one can perform the merging process step by step. Take e.g. M_1 and M_2 of \mathbb{M} and define $F_{1,2} \subseteq F_{\text{opt}}$ as the set of all edits that have been used to merge M_1 and M_2 so that they become twins in $G_{\text{opt}}[M]$, delete M_1 from \mathbb{M} and replace M_2 by $M_1 \cup M_2$ in \mathbb{M} and F_{opt} by $F_{\text{opt}} \setminus F_{1,2}$. Now all vertices in $M_1 \cup M_2$ have the same $\text{out}_{M_1 \cup M_2}$ -neighbors in $G[M] \triangle F_{1,2}$. Repeating this procedure reduces the size of \mathbb{M} by one element in each round and eventually terminates with $F(\mathbb{M}) := \bigcup_{i=1}^{l-1} F_{i,i+1} \subseteq F_{\text{opt}}$.

This strategy can be applied to all sets $\mathbb{M}^1 = \mathbb{M}, \mathbb{M}^2, \dots, \mathbb{M}^r$ that contain modules that are children of M in $\text{MDT}(G)$ and that have been merged to some new module $\mathcal{M}^1 = \mathcal{M}, \mathcal{M}^2, \dots, \mathcal{M}^r$, respectively. Hence, $\bigcup_{i=1}^r F(\mathbb{M}^i) \subseteq F_{\text{opt}}[M]$, where $F_{\text{opt}}[M]$ contains all (non-)edges $\{x, y\} \in F_{\text{opt}}$ with $x, y \in M$. Lemma 4.2 implies that $F_{\text{opt}}[M]$ is optimal in $G[M]$ and hence, the modules in $\mathbb{M}^1, \mathbb{M}^2, \dots, \mathbb{M}^r$ cannot be merged with fewer edits than in $F_{\text{opt}}[M]$. It follows that $\bigcup_{i=1}^r F(\mathbb{M}^i) = F_{\text{opt}}[M]$. Thus, there is always an optimal edit set F_{opt} that can be expressed by means of successively *pairwise* merge module edit, as done above. Algorithm 1 summarizes this approach in the form of pseudocode.

Algorithm 1 Simple Cograph Editing Heuristic.

Two functions, *get-module()* and *get-module-pair-edit()*, influence the practical efficiency, see text for details of their specification.

```

1: INPUT: A graph  $G = (V, E)$ ;
2: Compute  $\text{MD}(G)$ 
3: while  $M = \text{get-module}(\text{MD}(G))$  do
4:   if  $G[M]$  is a spider then
5:     Optimally edit  $G[M]$  to a cograph by application of EDP4 [LWGC11, LWGC12]
6:   else
7:     Let  $M_1, \dots, M_K$  be the children of  $M$  in  $\text{MDT}(G)$ , i.e.,  $\{M_1, \dots, M_K\} = \mathbb{P}_{\max}(G[M])$ .
8:      $(M_i, M_j, F_{i,j}) = \text{get-module-pair-edit}(\{M_1, \dots, M_K\})$ 
9:      $G \leftarrow (V, E(G) \triangle F_{i,j})$ 
10:   end if
11: end while
12: OUTPUT: The cograph  $G^*$ ;

```

Algorithm 1 contains two points at which the choice a particular module or a particular pair of modules affects performance and efficiency. First, the function *get-module()* returns a strong prime module that does not contain any other prime module and returns *false* if there is no such module, i.e., if G is a cograph. Second, subroutine *get-module-pair-edit()* extracts from $\{M_1, \dots, M_K\}$ a pair of modules. Ideally, these should satisfy the the following two conditions:

- (i) M_i and M_j have a minimum number of edits so that the $\text{out}_{M_i \cup M_j}$ -neighborhood in $G[M]$ becomes identical for all $x, y \in M_i \cup M_j$ among all pairs in $\mathbb{P}_{\max}(G[M])$, and

- (ii) additionally maximizes the number of removed P_4 's in $G[M]$ after applying these edits.

Lemma 5.1 *If $\text{get-module-pair-edit}()$ is an “oracle” that always returns a correct pairs M_i and M_j together with the respective edit set $F_{i,j}$ that is used to merged them and $\text{get-module}()$ returns an arbitrary strong prime module that does not contain any other prime module, then Alg. 1 computes an optimally edited cograph G_{opt} in $O(p\Lambda h(n)) \leq O(n^2 h(n))$ time, where p denotes the number of strong prime modules, $\Lambda = \max |\mathbb{P}_{\max}(M)|$ among all strong prime modules of G , and $h(n)$ is the cost for evaluating $\text{get-module-pair-edit}()$.*

Proof: The correctness of Algorithm1 follows directly from Lemma 4.8 and Theorem 4.10.

The modular decomposition $\text{MD}(G)$ can be computed in linear-time, see [CH94, DGM01, MS94, MS99, TCHP08]. Then, we have to resolve each of the p modules and in each step in the worst case all modules have to be merged stepwisely, resulting an effort of $O(|\mathbb{P}_{\max}|)$ merging steps in each iteration. Since $p \leq n$ and $\Lambda \leq n$ we obtain $O(n^2 h(n))$ as an upper bound. \square

In practice, the exact computation of the optimal editing pairs requires exponential effort. Practical heuristics for $\text{get-module-pair-edit}()$, however, can be implemented in polynomial time. A simple heuristic strategy to find those pairs can be established as follows: Mark all of the $O(\Lambda^2)$ pairs (M_i, M_j) where the set $\Gamma = N_G^{M_i} \triangle N_G^{M_j} \setminus (M_i \cup M_j)$ of distinct out_{M_i} - and out_{M_j} -neighbors that are not contained in M_i and M_j has minimum cardinality. Removing, resp., adding all edges xy with $x \in M_i \cup M_j$, $y \in \Gamma$ would yield a new module $M_i \cup M_j$ in the updated graph. Among all those marked pairs take the pair for a final merge that additionally removes a maximum number of induced P_4 's in the course of adjusting the respective out-neighborhoods. This amounts to an efficient method for detecting induced P_4 's. A detailed numerical evaluation of heuristics for cograph editing will be discussed elsewhere.

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